Meta-Theory of Generalised Algebraic Theories

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1 The Meta-Theory of Generalised Algebraic Theories

There are several ways of expressing the generalised algebraic theory of generalised algebraic theories. This document introduces one such which we call $MetaGAT$ in view of the fact that it can said to be a meta theory of generalised algebraic theories. We describe and discuss the relationship of the theory $MetaGAT$ to the theory of contextual categories.

2 Background

Contextual Categories provide algebraic representations of Generalised Algebraic Theories in the sense that to every Generalised Algebraic Theory U there is a contextual category $\mathbb{C}(U)$ such that algebras of the theory U are exactly structure preserving functors (contextual functors) from $\mathbb{C}(U)$ into the contextual category Fam of sets, families of sets and so on. We summarise this by saying that Contextual Categories provide the algebraic semantics of Generalised Algebraic Theories.

However, contextual categories contain structure that is in some sense redundant; this is in the same way that a Lawvere algebraic theory contains redundant structure compared to the otherwise equivalent notion, from universal algebra, of a clone. In both cases the redundancy is introduced in order to achieve a definition in which the algebraic structures are enrichments (informally speaking) of categories. Remove the redundant structure and there is no longer a category having surface visibility in the definitions.

The redundancy in the structure of a contextual category is exploited by Vladimir Voevodsky in his 'Subsystems And Regular quotients of C-systems' paper to give a Generalised Algebraic Theory of Contextual Categories. The significant point is that Voevodsky does this without introducing additional types whereas I had noted in my thesis that contextual categories could be described by a generalised algebraic theory with the introduction of additional equationally defined types to represent identity of morphisms. Voevodsky instead introduces an 's' operator on morphisms which in fact maps a morphism to a normal form representing that morphism as a section. By a section of type B, $Sec_{\mathbb{C}}(B)$, in a contextual category $\mathbb{C}$, I mean a morphism $f : A \to B$ in $\mathbb{C}$, where $A \triangleleft B$ and such that $f \circ p(A) = id_A$[1].

Compared to the theory of contextual categories, the theory $MetaGAT$ does not introduce redundant structure; it axiomatises the types and the terms of the theory directly and these correspond to the objects and the sections of a corresponding contextual category. The theory of contextual categories, on the other hand, axiomatises n-tuples of terms for any n; it is these that are represented by the morphisms of a contextual category.

3 Conclusion

(i) There is an isomorphism between the theory of contextual categories and the theory $MetaGAT$ in the category of existentially and identity enriched generalised algebraic theories.

(ii) There is no such isomorphism in the category of generalised algebraic theories.

(iii) The category of $MetaGAT$ algebras and the category of contextual categories are isomorphic.

[1]In my thesis the notation $Arr_{\mathbb{C}}(B)$ was used for what here we refer to as the sections and denote $Sec_{\mathbb{C}}(B)$

(iii) follows from (i) only because the contextual category Fam of sets, families of sets has existential quantification and identity types.

$MetaGAT$ algebras may have interpretations (and, therefore, applications) where Contextual Categories do not.

4 The $MetaGAT$ Theory

Sorts

The theory $MetaGat$ has two families of sort symbols representing, respectively, types and terms of a GAT.

Types of a GAT are represented by a family of sort symbols $\nabla_{i)_{i\geq 1}}$ introduced by rules as follows:

$$
\begin{array}{l}
\nabla_1 \text{ is a type} \\
x_1 \in \nabla_1 \vdash \nabla_2(x_1) \text{ is a type} \\
\vdots \\
x_1 \in \nabla_1, ...x_i \in \nabla_i(x_1, ...x_{i-1}) \vdash \nabla_{i+1}(x_1, ..x_i) \text{ is a type}
\end{array}
$$

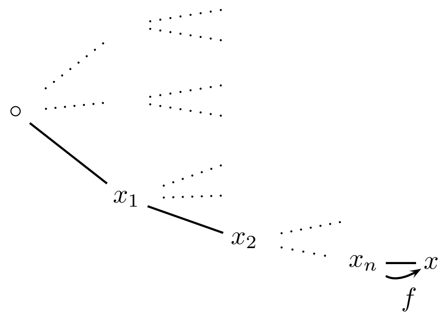
We can drop the subscripts from ∇ and omit the inessential variables so that we express as:

$$x_1 \in \nabla, ..x_n \in \nabla(x_{n-1}) \vdash \nabla(x_n) \text{ is a type} \tag{1}$$

Terms of a GAT are represented in $MetaGat$ by a family of sort symbols $\tau_{i)_{i\geq 1}}$. When x is a type at level i, $\tau_i(x)$ represents the terms of type x. As before, the subscripts and the inessential variables can be omitted and so, formally, each operator τ is introduced by:

$$x_1 \in \nabla, ...x_n \in \nabla(x_{n-1}), x \in \nabla(x_n) \vdash \tau(x) \text{ is a type} \tag{2}$$

We see, therefore, that a $MetaGAT$ algebra has a tree of types and for each type a set of terms. If, in some context, f is a term of type $\tau(x)$ then we picture it in the tree of types as an arrow leading to x from its predecessor node which, in some cases, may be the root of the tree. For example, in the context given in equation (2), if f is a term of type $\tau(x)$ then this is pictured so:



Operators

There are three families of operators in $MetaGAT$. In short they represent substitution denoted by *, weakening denoted by $\otimes$ and tautology (x in the context of x) which we call the diagonal and denote δ. We now describe the types of these operators in detail and a set of axioms that then complete the definition of $MetaGAT$.

Substitution - The * operators

A family of operators is introduced for substitution of terms into terms and types. Without ambiguity we can dispense with subscripts and use the symbol * throughout so that f^*t is denotes the substitution of term f into some term or type expression. To be more precise, if f is a term of type x, for some x, and if y is a type within a context which includes x, and if g is a term of type y, then f^*g is a term of type f^*y.

Formally, this is expressed by a collection of rules. In the simplest cases we have:

$$x \in \nabla, f \in \tau(x), y \in \nabla(x) \vdash f^*y \in \nabla \tag{3}$$

and

$$x \in \nabla, f \in \tau(x), y \in \nabla(x), g \in \tau(y) \vdash f^*g \in \tau(f^*y) \tag{4}$$

Rules (3) and (4) can be pictured like this:

$$\begin{tikzcd}
 & x \arrow[r, dash] \arrow[r, bend left, "g"] & y \\
\circ \arrow[ur, dash] \arrow[ur, bend left, "f"] \arrow[dr, dash] \arrow[dr, bend right, "f^*g"'] & & \\
 & f^*y &
\end{tikzcd}$$

More generally the rules for substitution are for $n \geq 0$ and $m \geq 0$:

$$x_1 \in \nabla, ...x_n \in \nabla(x_{n-1}), x \in \nabla(x_n), f \in \tau(x), y_1 \in \nabla(x), ...y_m \in \nabla(y_{m-1}), y \in \nabla(y_m) \\ \vdash f^*y \in \nabla(f^*y_m) \tag{5}$$

and

$$x_1 \in \nabla, ...x_n \in \nabla(x_{n-1}), x \in \nabla(x_n), f \in \tau(x), y_1 \in \nabla(x), ...y_m \in \nabla(y_{m-1}), y \in \nabla(y_m), \\ g \in \tau(y) \vdash f^*g \in \tau(f^*y) \tag{6}$$

In rule 6 all the context of rule 5 is given and then some (variable g is introduced). This is going to be a familiar pattern so we abbreviate by using double parentheses to reference an earlier context to be included. This enables us to represent rule 6 as follows:

$$((5)), g \in \tau(y) \vdash f^*g \in \tau(f^*y) \tag{7}$$

Rules (5) and (6) can be pictured like this:

$$\begin{tikzcd}
 & & & & x \arrow[r, dash] & y_1 & \cdots & y_m \arrow[r, dash] \arrow[r, bend right, "g"'] & y \\
\circ \arrow[r, dash] & x_1 & \cdots & x_n \arrow[ur, dash] \arrow[ur, bend left, "f"] \arrow[dr, dash] & & & & & \\
 & & & & f^*y_1 & \cdots & f^*y_m \arrow[r, dash] \arrow[r, bend right, "f^*g"'] & f^*y &
\end{tikzcd}$$

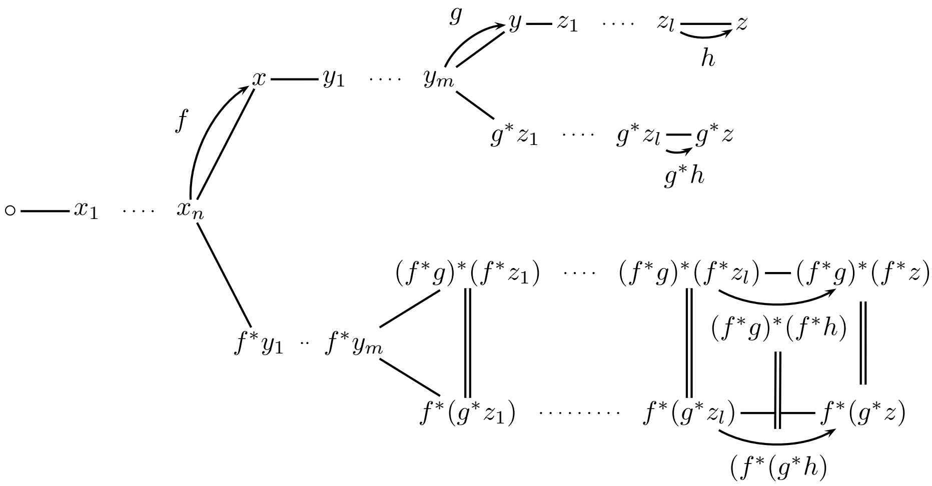
Substitution is subject to the rule that when both sides are defined $(f^*g)^*(f^*z) = f^*(g^*z)$.
The axioms, stated generally, are that, for $n \geq 0$,

$$x_1 \in \nabla, ...x_n \in \nabla(x_{n-1}), x \in \nabla x_n, f \in \nabla(x), y_1 \in \nabla(x_n), ...y_m \in \nabla(y_{m-1}), y \in \nabla y_m, g \in \nabla(y), \\ z_1 \in \nabla(y_m), ...z_l \in \nabla(z_{l-1}), z \in \nabla z_l \vdash (f^*g)^*(f^*z) = f^*(g^*z) \tag{8}$$

and

$$((8)), h \in \tau z \vdash (f^*g)^*(f^*h) = f^*(g^*h) \tag{9}$$

The combination of axioms (8) and (9) can be pictured like this:

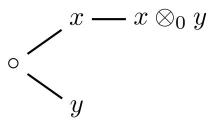


Weakening - The $\otimes$ operators

Introduction of context into a type or a term is represented by a family of operator symbols. In a completely formal presentation a second subscript is required but without ambiguity we can work with a notation with a family $\otimes_{i)_{i \geq 0}}$. In the simplest case:

$$x \in \nabla, y \in \nabla \vdash x \otimes_0 y \in \nabla(x) \tag{10}$$

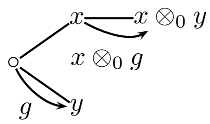
which we can picture like this:



and

$$x \in \nabla, y \in \nabla, g \in \tau \vdash x \otimes_0 g \in \tau(x \otimes_0 y) \tag{11}$$

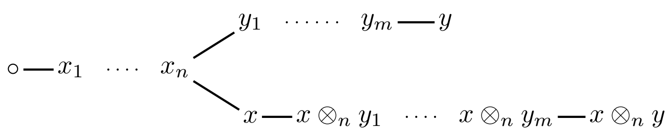
which we can picture like this:



The general rules for the introduction of the $\otimes_i$ family of operators are as follows. For $n \geq 0$ and $m \geq 0$, type weakening is introduced by:

$$x_1 \in \nabla, ...x_n \in \nabla(x_{n-1}), x \in \nabla(x_n), y_1 \in \nabla(x), ...y_m \in \nabla(y_{m-1}), y \in \nabla(y_m) \vdash x \otimes_n y_{\in} \nabla(x \otimes_n y_m) \tag{12}$$

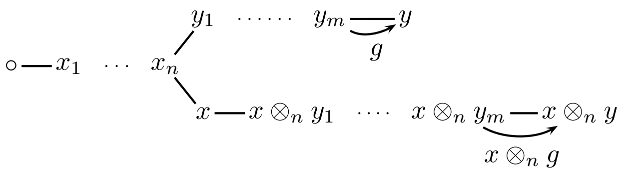
which can be pictured so:



The weakening of a term, in general, is introduced by:

$$((12)), g \in \tau(y) \vdash x \otimes_n g \in \tau(x \otimes_n y_m) \tag{13}$$

which we picture like this:



In what follows, we will drop all indices to the $\otimes$ symbol when there is no risk of ambiguity. Note that ambiguities can arise when the indices are omitted; we discuss these in a later section once were have completed this presentation of the theory.

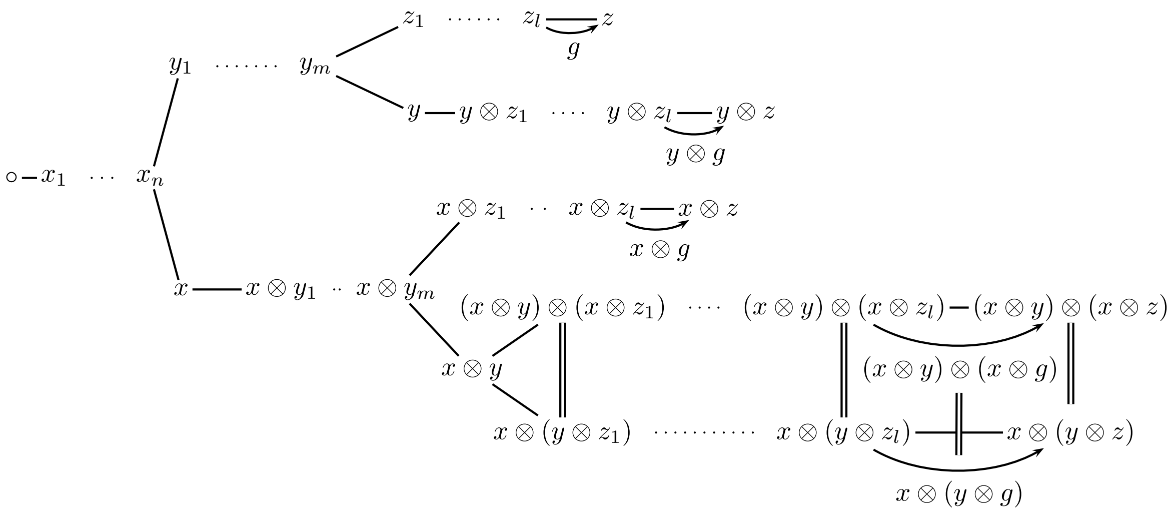
Weakening is subject to the following axioms when applied to weakened types, for $n \geq 0$, $m \geq 0$,:

$$x_1 \in \nabla, ...x_n \in \nabla(x_{n-1}), x \in \nabla(x_n), y_1 \in \nabla(x_n), ...y_m \in \nabla(y_{m-1}), y \in \nabla(y_m), z_1 \in \nabla(y_m), ...z_l \in \nabla(z_{l-1}),$$
$$z \in \nabla(z_l) \vdash (x \otimes y) \otimes (x \otimes z) = x \otimes (y \otimes z) \quad (14)$$

and the following when applied to weakened terms, for $n \geq 0$, $m \geq 0$,:

$$((14)), g \in \nabla(z_l) \vdash (x \otimes y) \otimes (x \otimes g) = x \otimes (y \otimes g) \quad (15)$$

The combination of axioms (14) and (15) can be pictured like this:



Axioms of Substitution and Weakening

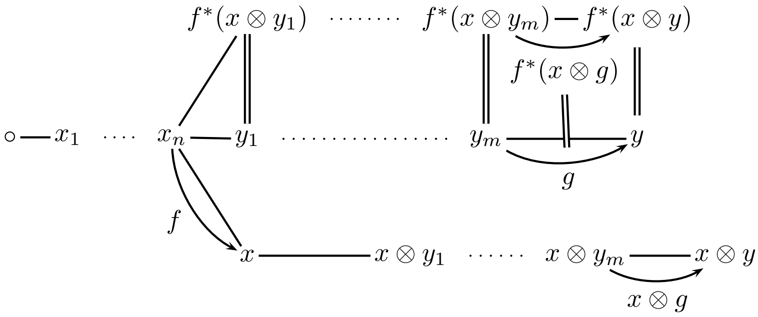
Two families of axioms govern behaviour of substitution into weakened types and terms.
Firstly, for $n \geq 0, m \geq 0$,

$$x_1 \in \nabla, ...x_n \in \nabla(x_{n-1}), x \in \nabla x_n, f \in \tau(x), y_1 \in \nabla(x_n), ...y_m \in \nabla(y_{m-1}), y \in \nabla y_m$$
$$\vdash f^*(x \otimes y) = y \quad (16)$$

and

$$((16)), g \in \tau(y) \vdash f^*(x \otimes g) = g \quad (17)$$

The combination of axioms (16) and (17) can be pictured like this:



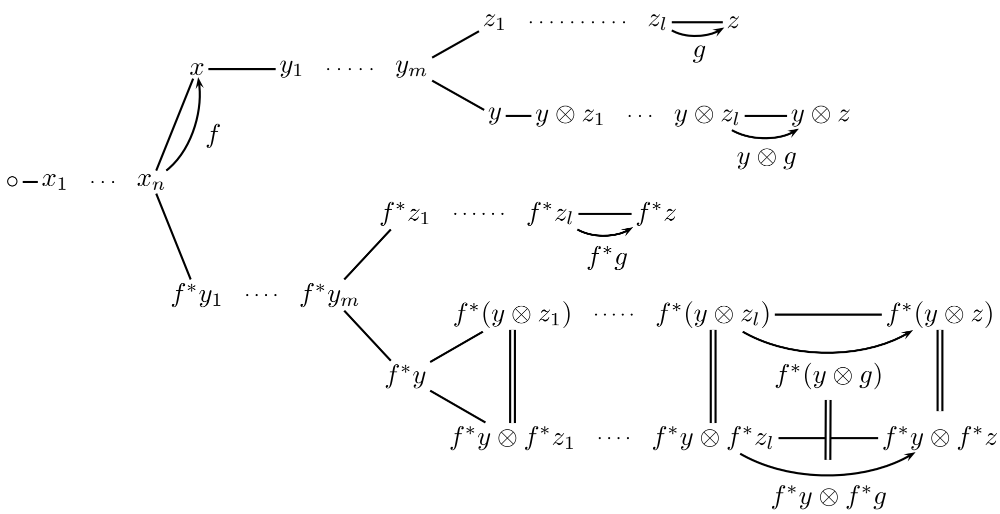
Secondly, $n \geq 0, m \geq 0, l \geq 0$:

$$x_1 \in \nabla, ...x_n \in \nabla(x_{n-1}), x \in \nabla(x_n), f \in \tau(x), y_1 \in \nabla(x), ...y_m \in \nabla(y_{m-1}), y \in \nabla(y_m)$$
$$z_1 \in \nabla(y_m), ...z_l \in \nabla(z_{l-1}), z \in \nabla(z_l) \vdash f^*y \otimes f^*z = f^*(y \otimes z) \quad (18)$$

and

$$((18)), g \in \tau(z) \vdash f^*y \otimes f^*g = f^*(y \otimes g) \quad (19)$$

The combination of axioms (18) and (19) can be pictured like this:



In addition there are axioms governing the substitution of weakened terms into weakened terms. For $n \geq 0, m \geq 0, l \geq 0$:

$$x_1 \in \nabla, ...x_n \in \nabla(x_{n-1}), x \in \nabla(x_n), y_1 \in \nabla(x_n), ...y_m \in \nabla(y_{m-1}), y \in \nabla(y_m), g \in \tau(y)$$
$$z_1 \in \nabla(y), ...z_l \in \nabla(z_{l-1}), z \in \nabla z_l \vdash (x \otimes g)^*(x \otimes z) = x \otimes (g^*z) \quad (20)$$

and

$$((20)), h \in \nabla(z) \vdash (x \otimes g)^*(x \otimes h) = x \otimes (g^*h) \quad (21)$$

Axioms (20) and (21) can be pictured like this:

$$\begin{tikzcd}
 & & & & & & & y \arrow[r, no head] & z_1 & \cdots & z_l \arrow[r, no head] \arrow[r, bend right, "h"'] & z \\
 & & & & y_1 & \cdots & y_m \arrow[ur, no head] \arrow[ur, bend right, "g"'] \arrow[dr, no head] \\
 & & & & & & & g^*z_1 & \cdots & g^*z_l \arrow[r, no head] \arrow[r, bend right, "g^*h"'] & g^*z \\
\circ \arrow[r, no head] & x_1 & \cdots & x_n \arrow[uur, no head] \arrow[ddr, no head] \\
 & & & & & & & & x \otimes y \arrow[r, no head] & x \otimes z_1 & \cdots & x \otimes z_l \arrow[r, no head] \arrow[r, bend right, "x \otimes h"'] & x \otimes z \\
 & & & & x \arrow[r, no head] & x \otimes y_1 & \cdots & x \otimes y_m \arrow[ur, no head] \arrow[ur, bend left, "x \otimes g"] \arrow[r, no head] \arrow[dr, no head] & x \otimes g^*z_1 \arrow[d, equals] & \cdots & x \otimes g^*z_l \arrow[r, no head] \arrow[r, bend right, "(x \otimes g)^*h"'] \arrow[d, equals] & x \otimes g^*z \arrow[d, equals] \\
 & & & & & & & & (x \otimes g)^*(x \otimes z_1) & \cdots & (x \otimes g)^*(x \otimes z_l) \arrow[r, no head, "\Vert" description] \arrow[r, bend right, "(x \otimes g)^*(x \otimes h)"'] & (x \otimes g)^*(x \otimes z)
\end{tikzcd}$$

Diagonal - The δ Operators

The diagonal (x, in the context of x) is in the simplest case introduced by this rule:

$$x \in \nabla \vdash \delta(x) \in x \otimes x \tag{22}$$

which is pictured like this:

$$\begin{tikzcd}
\circ \arrow[r, no head] & x \arrow[r, no head] \arrow[r, bend right, "\delta x"'] & x \otimes x
\end{tikzcd}$$

More generally, for $n \geq 1$:

$$x_1 \in \nabla, ...x_n \in \nabla(x_{n-1}) \vdash \delta(x_n) \in x_n \otimes x_n \tag{23}$$

which is pictured like this:

$$\begin{tikzcd}
\circ \arrow[r, no head] & x_1 & \cdots & x_n \arrow[r, no head] \arrow[r, bend right, "\delta x_n"'] & x_n \otimes x_n
\end{tikzcd}$$

The diagonal δ is subject to four axiom schemes.

Firstly, axioms for substituting the diagonal into weakenings, for $n \geq 0$:

$$x_1 \in \nabla, ...x_n \in \nabla(x_{n-1}), x \in \nabla(x_n), y_1 \in \nabla(x), ...y_m \in \nabla(y_{m-1}), y \in \nabla(y_m) \vdash \delta(x) * (x \otimes y) = y \tag{24}$$

and

$$((24)), g \in \nabla(y) \vdash \delta(x) * (x \otimes g) = g \tag{25}$$

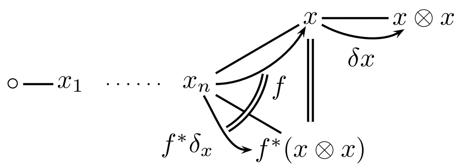
Axioms (24) and (25) can be pictured like this:

$$\begin{tikzcd}
 & & & & & y_1 \arrow[d, equals] & \cdots & y_m \arrow[r, no head] \arrow[r, bend right, "g"'] \arrow[d, equals] & y \arrow[d, equals] \\
\circ \arrow[r, no head] & x_1 & \cdots & x_n \arrow[r, no head] & x \arrow[ur, no head] \arrow[r, no head] \arrow[dr, no head] \arrow[dr, bend right, "\delta_x"'] & \delta_x^*(x \otimes y_1) & \cdots & \delta_x^*(x \otimes y_m) \arrow[r, no head, "\Vert" description] \arrow[r, bend right, "\delta_x^*(x \otimes g)"'] & \delta_x^*(x \otimes y) \\
 & & & & & x \otimes x \arrow[r, no head] & x \otimes y_1 & \cdots & x \otimes y_m \arrow[r, no head] \arrow[r, bend right, "x \otimes g"'] & x \otimes y
\end{tikzcd}$$

Secondly axioms for substituting into a diagonal:

$$x_1 \in \nabla, ...x_n \in \nabla(x_{n-1}), x \in \nabla(x_n), f \in \tau(x) \vdash f^*\delta(x) = f \tag{26}$$

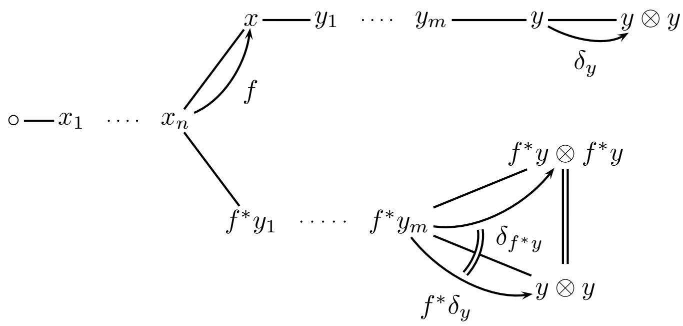
which is pictured like this:



and

$$x_1 \in \nabla, ...x_n \in \nabla(x_{n-1}), x \in \nabla(x_n), f \in \tau(x), y_1 \in \nabla(x), ...y_m \in \nabla(y_{m-1}), y \in \nabla(y_m)$$
$$\vdash f^*\delta(y) = \delta(f^*y) \tag{27}$$

Axiom 27 can be pictured like this:



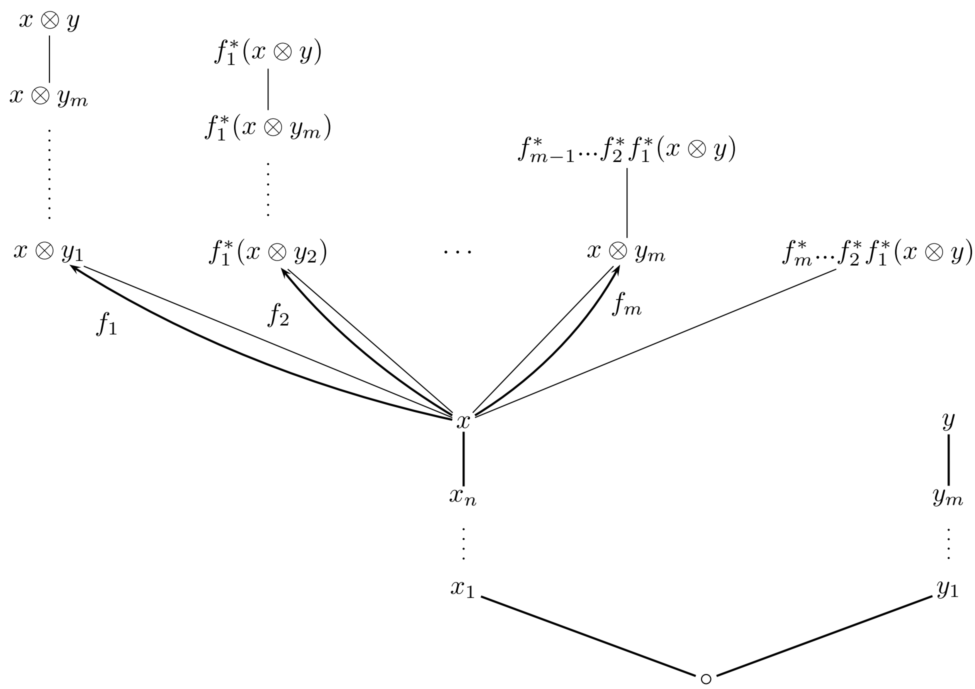
This completes the definition of the generalised algebraic theory $MetaGAT$.

5 Relating Contextual Categories and $MetaGAT$ algebras

5.1 Constructing a Contextual Category from a $MetaGAT$ algebra

From a $MetaGAT$ algebra $\mathcal{A}$ we can construct a corresponding contextual category whose sections are the terms of the $MetaGAT$ algebra. The tree of objects of the contextual category is the tree of types of the $MetaGAT$ algebra[2]. The morphisms $Hom(A, B)$ where $1 \triangleleft A_1 ... A_n \triangleleft A$ and $1 \triangleleft B1 ... \triangleleft B_m \triangleleft B$ are m-tuples $f_1, ...f_m, f$ of terms of $\mathcal{A}$ which are typed as shown in the following diagram:

[2] A root to the tree is required and therefore formally appended as this is not explicit as a type in $MetaGAT$.



5.2 Interpreting the Theory of Contextual Categories in $MetaGat$

Vladimir Voevodsky in this C-systems paper shows how the theory of contextual categories can be expressed as a generalised algebraic theory. The construction outlined in section 5.1 can be expressed as an interpretation of the generalised algebraic theory of contextual categories into the $MetaGAT$ theory enriched by existential types and the absolute. Absolute is required for the interpretation of the terminal object of the contextual category. Existential types over terms are required so that the morphism sort (Hom) of the contextual category can be interpreted as follows. In the context:

$$x_1 \in Ob_1, ...x_n \in Ob_n(x_{n-1}), x \in Ob_{n+1}(x_n), y_1 \in Ob_1, ...y_m \in Ob_m(y_{m-1}), y \in Ob_{m+1}(y_m) \quad (28)$$

the type $Hom(x, y)$ is interpreted by:

$$\sum_{f_1 \in \tau(x \otimes y_1)} \sum_{f_2 \in \tau(f_1^*(x \otimes y_2))} \cdots \sum_{f_m \in \tau(f_{m-1}^*(f_{m-2}^* \cdots (f_1^*(x \otimes y_m))\ldots))} \tau(f_m^*(f_{m-1}^* \cdots (f_1^*(x \otimes y))\ldots)) \quad (29)$$

5.3 Interpreting $MetaGat$ in the Theory of Contextual Categories

If the theory of contextual categories is enriched by identity and existential types over the Hom types then the construction of a $MetaGAT$ algebra from the sections of the contextual category can expressed as an interpretation of the theory $MetaGAT$ in the enriched theory of contextual categories. The key to the interpretation is that terms of $MetaGAT$ are interpreted as sections of the contextual category and this can be expressed as a type in the enriched theory as follows. In the context

$$x_1 \in Ob_1, ...x_n \in Ob_n(x_{n-1}), x \in Ob_{n+1}(x_n)$$

$Sections(x)$ is the type

$$\sum_{f \in Hom(x_n, x)} Id(f \circ p(x), id_{x_n})$$